Exact solutions of the Schrodinger equation $(-d/dx^2 + x^2 + \lambda x^2/(1 + gx^2))\psi(x) = E\psi(x)$

1982 J. Phys. A: Math. Gen. 15 1217

(http://iopscience.iop.org/0305-4470/15/4/024)

J. Phys. A: Math. Gen. **15** (1982) 1217–1226. Printed in Great Britain

Exact solutions of the Schrödinger equation $\left(-\frac{d^2}{dx^2}+x^2+\frac{\lambda x^2}{1+gx^2}\right)\psi(x)=E\psi(x)$

R R Whitehead†, A Watt†, G P Flessas† and M A Nagarajan‡

† Department of Natural Philosophy, The University, Glasgow G12 8QQ, UK

‡ Daresbury Laboratory, Daresbury, Warrington, UK

Received 9 September 1981, in final form 12 November 1981

Abstract. We prove the existence of a class of exact eigenvalues and eigenfunctions of the Schrödinger equation for the potential

$$x^2+\frac{\lambda x^2}{1+gx^2}$$

when certain algebraic relations between λ and g hold. Some of the properties of these solutions are discussed. It is shown that in a certain sense they may be regarded as Sturmians for the Schrödinger equation with the potential

$$x^2-\frac{\lambda}{g}\frac{1}{1+gx^2}.$$

1. Introduction

The purpose of this work is to investigate the Schrödinger equation

$$y''(x)+\left(E-x^2-\frac{\lambda x^2}{1+gx^2}\right)y(x)=0 \qquad g>0 \quad -\infty<x<\infty \tag{1.1}$$

where E is the eigenparameter and λ is a real number. This equation appears in several areas of physics. In field theory it provides a simple zero-dimensional model possessing a non-polynomial Lagrangian (Biswas *et al* 1973). In laser physics it arises out of the Fokker–Planck equation for a single-mode laser (Risken and Vollmer 1967, Haken 1970).

Equation (1.1) has been studied numerically by Mitra (1978) and by Bessis and Bessis (1980) using variational and perturbation methods in the case $\lambda>0$. Recently (Flessas 1981) the existence of exact solutions, valid when certain algebraic relations between λ and g hold, and which can, moreover, be written in closed form, has been demonstrated.

In this paper we will begin by proving the existence of a class of solutions which, when λ and g are suitably chosen, may be written as terminating polynomials. In view of the restrictions on λ and g, these do not constitute the complete set of solutions; indeed, for arbitrary λ and g there may well be no such solutions. In § 2 we derive the required results using the theory of linear differential equations, while in § 3 we examine the solutions so obtained by an algebraic method and show that, from this point of view, their existence is a consequence of certain algebraic peculiarities of the Hamiltonian of (1.1). In § 4 we show that the solutions in question are the Sturmians of a related differential equation, and in § 5 we summarise the results.

2. The set of exact solutions

2.1. Treatment of the Schrödinger equation

The substitutions

$$x^2 = t \qquad y(x) = P(t)\exp(-t/2) \tag{2.1}$$

transform (1.1) into

$$P''(t) + \frac{-4gt^2 + t(2g-4)+2}{4t(1+gt)}P'(t) + \frac{[t(Eg-\lambda-g)+E-1]t}{4t^2(1+gt)}P(t) = 0. \tag{2.2}$$

The indicial equation of (2.2) has roots 0 and $\frac{1}{2}$ which correspond respectively to the even-parity and odd-parity states; these two types form a fundamental system for (2.2). In what follows we will confine ourselves to the even-parity case. The treatment of the odd-parity states is identical, and the relevant formulae will be given at the end of § 2.

In the usual way we have that

$$P(t) = \sum_{k=0}^{\infty} C_k t^k \tag{2.3}$$

is an exact solution of (2.2) convergent for $t \in [0, \infty)$, since the only singularities of (2.2) are at $t = 0$ and $t = \infty$. Inserting (2.3) into (2.2) we get the three-term recurrence relation

$$C_{k+1}\beta_k + C_k\gamma_k + C_{k-1}\delta_k = 0 \qquad k = 0, 1, 2, \ldots \qquad C_{-1} = 0 \tag{2.4}$$

where

$$\beta_k = 2(k+1)(2k+1) \qquad \gamma_k = E - 1 + k(4kg - 2g - 4)$$
$$\delta_k = Eg - \lambda - g - 4(k-1)g. \tag{2.5}$$

From (2.4) it follows immediately that

$$\frac{C_k}{C_{k-1}} = \cfrac{-\delta_k}{\gamma_k - \cfrac{\beta_k\delta_{k+1}}{\gamma_{k+1} - \cfrac{\beta_{k+1}\delta_{k+2}}{\gamma_{k+2} - \ddots}}}. \tag{2.6}$$

In closed form C_n is given by

$$C_n = \frac{(-1)^n D_n}{(2n)!} \qquad n = 0, 1, 2, \ldots \qquad C_0 = 1 \tag{2.7}$$

with

$$
D_n = \begin{vmatrix}
E-1 & 2 & 0 & 0 & & \\
Eg-\lambda-g & E-1+2g-4 & 2\cdot 2\cdot 3 & 0 & & \\
0 & \begin{pmatrix} Eg-\lambda-g \\ -1\cdot 4\cdot g \end{pmatrix} & E-1+2(6g-4) & 2\cdot 3\cdot 5 & \ddots & \\
0 & 0 & \begin{pmatrix} Eg-\lambda-g \\ -2\cdot 4\cdot g \end{pmatrix} & E-1+3(10g-4) & \ddots & \\
\vdots & \vdots & & \ddots & \begin{pmatrix} E-1+(n-2) \\ \cdot(4gn-10g-4) \end{pmatrix} & 2(n-1)(2n-3) \\
 & & & & \begin{pmatrix} Eg-\lambda-g \\ -(n-2)4g \end{pmatrix} & \begin{pmatrix} E-1+(n-1) \\ \cdot[4g(n+1)-10g-4] \end{pmatrix}
\end{vmatrix} \tag{2.8}
$$

The D_n satisfy the difference equation

$$
\begin{aligned}
&D_n = \gamma_{n-1} D_{n-1} - \beta_{n-2}\delta_{n-1} D_{n-2} \qquad n = 2, 3, \ldots \\
&D_0 = 1 \qquad D_1 = E - 1.
\end{aligned} \tag{2.9}
$$

In order that not all the C_n vanish identically it is necessary that

$$D_\infty = 0. \tag{2.10}$$

The condition (2.10), while ensuring the existence of the C_n and the convergence of the continued fraction (2.6), does *not*, however, necessarily imply that $P(t)$ increases sufficiently slowly as $t \to \infty$ for $y(x) = P(t)\exp(-t/2)$ to remain normalisable. This is, of course, well known in the usual treatment of the one-dimensional harmonic oscillator, which in fact corresponds to $\lambda = 0$ in (1.1). In that case,as here, the infinite series (2.3) exists but is non-normalisable and so not acceptable as a physical solution. One should investigate the precise form of the limit $C_{n+1}/C_n|_{n\to\infty}$ and from it infer the behaviour of $P(t)$ for $t \to \infty$. In our case the basic relation (2.4) gives trivially $C_{n+1}/C_n|_{n\to\infty} = 0$, a fact already known from the convergence of (2.3); indeed, we have $|(C_{n+1}/C_n)t|_{n\to\infty} < 1$, $t \in [0, \infty)$ and so again $C_{n+1}/C_n|_{n\to\infty} = 0$. In the absence of a more detailed expression for this limit, (2.10) cannot guarantee that the infinite-series solutions are physical. They may be but we cannot yet say. Instead, as in the treatment of the harmonic oscillator, we look for solutions in which the series (2.3) terminates.

2.2. Polynomial-type solutions

Terminating solutions may be obtained by requiring that $C_N = 0$ for $N > n+1$. Then we see from (2.6) and (2.7) that we need the following to hold:

$$Eg - \lambda - g = 4gn \qquad n = 1, 2, 3, \ldots \tag{2.11}$$

$$D_{n+1} = 0. \tag{2.12}$$

For $n = 1, 2$ equations (2.11) and (2.12) give rise to the particular cases found by Flessas (1981).

It is worth noticing here that, although the potential under discussion is markedly different from that of the doubly anharmonic oscillator (Singh *et al* 1980) or that of the generalised anharmonic oscillator (Flessas and Das 1980, Magyari 1981), it shares with them the presence of polynomial-type solutions. An important difference, however, is that, while in the case of the anharmonic oscillator the existence of the polynomials has to be examined for each n separately (Singh *et al* 1980), we have in the present case the following theorem.

Theorem 2.1. Conditions (2.11) and (2.12) are sufficient for the existence of

$$P_n(t)=\sum_{k=0}^{n} C_k^{(n)}t^k \qquad n=1,2,\ldots. \tag{2.13}$$

Proof. To prove this it is evidently sufficient to show that the algebraic equation of degree $n+1$ obtained by putting (2.11) into (2.12) has real roots in $\lambda(g)$ or $g(\lambda)$. Writing $z=\lambda/g$ we have

$$D_{n+1}=\begin{vmatrix} 4n+z & 2 & 0 & & & \\ 4gn & 4n+z+2g-4 & 2\cdot 2\cdot 3 & \ddots & & \\ 0 & 4gn-1\cdot 4\cdot g & 4n+z+2(6g-4) & \ddots & \ddots & \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & 2n(2n-1) \\ \vdots & \vdots & & & 4gn-(n-1)4g & 4n+z+n(4g(n+2)-10g-4) \end{vmatrix}=0. \tag{2.14}$$

For $\lambda=0$ (1.1) reduces to the Schrödinger equation for the one-dimensional harmonic oscillator, and since its eigenfunctions are of the form (2.13) we expect $\lambda=0$ ($z=0$) to be one of the roots of (2.14). Indeed, this is so as may be seen from (2.14) by setting $z=0$ and then subtracting from each row the previous one multiplied by g. Then

$$D_{n+1}|_{z=0}=\begin{vmatrix} 4n & 2 & 0 & & \\ 0 & 4n-4 & 2\cdot 2\cdot 3 & \ddots & \\ 0 & 0 & 4n-2\cdot 4 & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & & 0 & 4n-n\cdot 4 \end{vmatrix} \tag{2.15}$$

and the vanishing of the last row proves the assertion.

To prove the reality of the other roots it is only necessary to notice that D_{n+1} is tri-diagonal with real positive off-diagonal elements. Then D_{n+1} is identical with a symmetric determinant $\mathscr{D}_{n+1}$ whose elements are

$$\mathscr{D}_{n+1}(i,i)=D_{n+1}(i,i)$$
$$\mathscr{D}_{n+1}(i,i+1)=\mathscr{D}_{n+1}(i+1,i)=[D_{n+1}(i+1,i)D_{n+1}(i,i+1)]^{1/2}$$

and the reality of the roots is apparent.

For arbitrary $g>0$ and any n we can find from (2.14) $n+1$ real values of λ, which will depend on g, and thence the $n+1$ energies from (2.11). The corresponding polynomials $P_n(t)$ may then be deduced from (2.7), (2.8). This completes the proof of theorem 2.1.

A further result of physical importance can be established immediately:

Theorem 2.2. The non-zero roots of (2.14) are all negative.

Proof. The sub-determinants

$$D_{n+1}^{(1)} = 4n - (-z) \qquad D_{n+1}^{(2)} = \begin{vmatrix} 4n-(-z) & 2 \\ 4gn & 4n+2g-4-(z) \end{vmatrix} \qquad \ldots$$

constitute a Sturm sequence (the roots of $D_{n+1}^{(i)}$ separate those of $D_{n+1}^{(i+1)}$). It may be shown that, as is well known in numerical analysis (Gantmacher 1959), the number of agreements in sign between successive members of the sequence

$$1 \qquad D_{n+1}^{(1)}(x) \qquad D_{n+1}^{(2)}(x) \qquad \ldots \qquad D_{n+1}(x)$$

is equal to the number of roots of $D_{n+1}(x)$ which are greater than or equal to x. Setting $-z = x = 0$ (2.15) shows that there are $n+1$ sign agreements and thus the $n+1$ values of $-z$ are all greater than zero. So for fixed $g > 0$ polynomial solutions exist only for $\lambda < 0$.

2.3. *Odd-parity solutions*

The foregoing results hold also for the odd-parity solutions but with the following differences. In place of (2.3) we have

$$H(t) = \sum_{k=0}^{\infty} d_k t^{k+1/2}$$

and the d_k satisfy a three-term recurrence relation like (2.4) but with coefficients

$$\beta'_k = 2(k+1)(2k+3) \qquad \gamma'_k = E - 1 + 2(2k+1)(gk-1)$$

$$\delta'_k = Eg - \lambda - g - 2(2k-1)g.$$

The important relation (2.11) is replaced by

$$Eg - \lambda - g = 2g(2n+1) \qquad n = 1, 2, 3, \ldots$$

and $\lambda(g)$ is determined by solving the analogue of (2.14).

3. The algebraic method

In this section we treat the same problem as before by introducing a basis and proceeding algebraically. The object here is to bring out the particular features of the differential equation (1.1) that allow the polynomial-type solutions to exist and which are not readily apparent from the treatment of § 2.

3.1. *The oscillator basis*

The natural basis for an algebraic investigation of (1.1) consists of the eigenstates $|n\rangle$ of the harmonic oscillator (defined without the usual factor of $\frac{1}{2}$)

$$H_0 = p^2 + x^2 \tag{3.1}$$

$$H_0|n\rangle = (2n+1)|n\rangle = \varepsilon_n|n\rangle. \tag{3.2}$$

In this basis the Hamiltonian

$$H = H_0 + \lambda x^2/(1+gx^2) \tag{3.3}$$

is an infinite-dimensional matrix and has no readily discernible structure (patterns of zero elements, for example) that would lead one to suspect the existence of the solutions in question. From (2.13) we see that these solutions are in fact confined to the $(N+1)$-dimensional subspaces $|0\rangle, |2\rangle, \ldots, |2N\rangle$ for even parity and $|1\rangle, |3\rangle, \ldots, |2N+1\rangle$ for odd parity. For a given N there are $N+1$ such solutions, one of which corresponds to $\lambda = 0$ and is just, according to (2.11), the state $|N\rangle$ (even) or $|N+1\rangle$ (odd).

We begin by investigating the matrix elements of the interaction term in (3.3) in the oscillator basis. We write

$$[x^2/(1+gx^2)]|n\rangle = \begin{cases} \sum_{i=0}^{\infty} a_{ni}|i\rangle & n, i \text{ even} \\ \sum_{i=1}^{\infty} a_{ni}|i\rangle & n, i \text{ odd} \end{cases} \tag{3.4}$$

and seek to establish certain facts about the a_{ni} in a rather general way (in particular without introducing any explicit expressions and thereby restricting the analysis to this one case). Henceforth we will consider the even-parity case only; odd-parity states are treated identically by simply taking odd indices and state labels. The following important theorem holds:

Theorem 3.1. The coefficients a_{ni} obey the relations

$$\frac{a_{n0}}{a_{m0}} = \frac{a_{n2}}{a_{m2}} = \ldots = \frac{a_{nn-2}}{a_{mn-2}} \qquad n > 2 \quad m > n. \tag{3.5}$$

Proof. Equation (3.4) may be rewritten as

$$\sum_{i=0}^{\infty} a_{ni}(1+gx^2)|i\rangle = x^2|n\rangle \qquad n, i \text{ even}$$

in consequence of the commutativity of x^2 and $1+gx^2$. Taking matrix elements of both sides with every (even) oscillator state in turn leads to the infinite-matrix equation

$$\begin{pmatrix} \gamma_0 & \delta_0 & 0 & 0 & \cdots \\ \delta_0 & \gamma_2 & \delta_2 & 0 & \cdots \\ 0 & \delta_2 & \gamma_4 & \delta_4 & \\ 0 & 0 & \ddots & \ddots & \ddots \\ \vdots & \vdots & & & \end{pmatrix} \begin{pmatrix} a_{n0} \\ a_{n2} \\ a_{n4} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{n-2} \\ \alpha_n \\ \beta_n \\ 0 \\ \vdots \end{pmatrix} \tag{3.6}$$

where α, β and γ, δ are the oscillator matrix elements of x^2 and $1+gx^2$ respectively. The structure of (3.6) follows directly from the fact that x^2 and $1+gx^2$ are both tri-diagonal in the oscillator basis. It is easy to show, by partitioning (3.6), that

$$\begin{pmatrix} a_{n0} \\ a_{n2} \\ \vdots \\ a_{nn-2} \end{pmatrix} = (\beta_{n-2} - \delta_{n-2}a_{nn})X \tag{3.7}$$

where

$$X=\begin{pmatrix} \gamma_0 & \delta_0 & 0 & \dots \\ \delta_0 & \gamma_2 & \delta_2 & \\ 0 & \ddots & \ddots & \ddots \\ \vdots & & & \gamma_{n-2} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{3.7a}$$

and that, with the same partitioning and $m>n$,

$$\begin{pmatrix} a_{m0} \\ \vdots \\ a_{mn-2} \end{pmatrix}=-\delta_{n-2}a_{mn}X. \tag{3.8}$$

Then the truth of theorem 3.1 is evident on comparing (3.8) with (3.7).

3.2. *The solutions*

Having ascertained the validity of (3.5), we observe that, if there exists a vector $(y_0, y_2, \dots, y_{n-2})^{\mathrm{T}}$ with the property that

$$\begin{pmatrix} \varepsilon_0+\lambda a_{00} & \lambda a_{20} & \lambda a_{40} & \dots \\ \lambda a_{20} & \varepsilon_2+\lambda a_{22} & \lambda a_{42} & \dots \\ \lambda a_{40} & \lambda a_{42} & \ddots & \\ \vdots & \vdots & & \varepsilon_{n-2}+\lambda a_{n-2n-2} \end{pmatrix}\begin{pmatrix} y_0 \\ y_2 \\ \vdots \\ y_{n-2} \end{pmatrix}=E\begin{pmatrix} y_0 \\ y_2 \\ \vdots \\ y_{n-2} \end{pmatrix} \qquad n \text{ even}>2 \tag{3.9}$$

and which is at the same time orthogonal to X (3.7a), then E will be an eigenvalue of the full Hamiltonian and the augmented vector $(y_0, y_2 \dots; y_{n-2}, 0, 0, \dots)^{\mathrm{T}}$ will be the corresponding eigenvector. The orthogonality constraint implies that such vectors will only exist, if at all, for certain values of λ. We now assert that the polynomial solutions of § 2 are of just this kind.

To verify the truth of this assertion we rewrite the eigenvalue problem as

$$\lambda x^2 y(x)=(1+gx^2)(E-p^2-x^2)y(x).$$

Then, expanding $y(x)$ in the oscillator functions, we get an infinite set of linear equations for the expansion coefficients. We seek solutions for which all the coefficients vanish except $y_0, \dots, y_{n-2}$. This leads directly to the consistency condition (cf (2.11))

$$\lambda=(E-\varepsilon_{n-2})g \tag{3.10}$$

and the finite-dimensional secular equation

$$\begin{pmatrix} 1+g\alpha_0 & g\beta_0 & 0 & 0 \\ g\beta_0 & 1+g\alpha_2 & g\beta_2 & \\ 0 & \ddots & \ddots & \ddots \\ 0 & & & 1+g\alpha_{n-2} \end{pmatrix}\begin{pmatrix} \varepsilon_0-\varepsilon_{n-2} & 0 & 0 \\ \vdots & \varepsilon_2-\varepsilon_{n-2} & 0 \\ & & \ddots \\ 0 & 0 & \varepsilon_{n-2}-\varepsilon_{n-2} \end{pmatrix}\begin{pmatrix} y_0 \\ \vdots \\ y_{n-2} \end{pmatrix}$$

$$=\frac{\lambda}{g}\begin{pmatrix} y_0 \\ \vdots \\ y_{n-2} \end{pmatrix} \tag{3.11}$$

where we have used the fact that $\delta_i = g\beta_i$ and $\gamma_i = 1 + g\alpha_i$. Equation (3.11) will be seen with a little thought to be essentially the same as (2.14), but here its properties are more transparent. The matrix on the left of (3.11) is the product of a symmetric matrix and a negative diagonal matrix. It thus has $n/2$ real roots and its eigenvectors are orthogonal with respect to the (indefinite) metric

$$D = \begin{pmatrix} \varepsilon_0 - \varepsilon_{n-2} & & 0 \\ & \ddots & \\ 0 & & \varepsilon_{n-2} - \varepsilon_{n-2} \end{pmatrix}. \tag{3.12}$$

Moreover, as is evident from (3.11), one root is always zero and the corresponding eigenvector is $|n-2\rangle$. Putting $n - 2 = 2N$ to restore the labelling used at the beginning of this section and in § 2, the zero root has eigenvector $|2N\rangle$ and its energy is $E = \varepsilon_{2N}$. The other N roots are linear combinations of the oscillator states $|0\rangle, |2\rangle, \ldots, |2N-2\rangle$.

It now remains only to show that the eigenvectors of (3.11) are indeed orthogonal to the vector $\boldsymbol{X}$ (3.7a). If we write (3.11) as

$$TDy = \frac{\lambda}{g} y$$

we have

$$Dy = \frac{\lambda}{g} T^{-1} y$$

and since the last element of D (3.12) is zero it is evident that $T^{-1}y$ is orthogonal to the vector $(0, 0, \ldots, 0, 1)^{\mathrm{T}}$ in which there are $n/2 - 1$ zeros. But $(0, 0, \ldots, 0, 1)^{\mathrm{T}} T^{-1} = \text{constant} \times \boldsymbol{X}^{\mathrm{T}}$ in view of the relationships between α, β and γ, δ, and the required orthogonality is verified.

From the generality of the above discussion we can deduce immediately that any Hamiltonian

$$H = H_0 + \lambda (a + bG)^{-1} G$$

where the operator G is tri-diagonal in the basis in which H_0 is diagonal will possess analogous systems of solutions.

4. The solutions as Sturmians

We show here that the solutions found in §§ 2 and 3 have properties analogous to those of the so-called Sturmians (Rotenberg 1962). Equation (1.1) can be rearranged as

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 - \frac{\lambda}{g} \frac{1}{1 + gx^2}\right) y(x) = \left(E - \frac{\lambda}{g}\right) y(x). \tag{4.1}$$

Using condition (2.11) and the notation of § 3 (cf (3.10)) the right-hand side is simply $\varepsilon_{n-2} y(x)$. To calculate the terminating polynomial-type solutions, therefore, we solve (4.1) by varying λ until the boundary conditions $y(\pm\infty) = 0$ are satisfied, always keeping the coefficient of $y(x)$ on the right-hand side equal to ε_{n-2}. But this is just the prescription for calculating Sturmian functions—the normal eigenparameter is held fixed while the strength of the interaction is varied.

Consider two such functions $y_1(x)$, $y_2(x)$ obtained with the same ε_{n-2}. It can easily be shown from (4.1) that

$$\int_{-\infty}^{\infty} y_1(x) \frac{1}{1+gx^2} y_2(x) \, dx = 0 \tag{4.2}$$

which is the form taken here by the most important property of the Sturmians. Their other important property of completeness has here to be modified slightly. Our functions, corresponding to a given ε_{n-2}, are complete in the space spanned by the oscillator functions $|0\rangle, |2\rangle, \ldots, |n-2\rangle$ (for even parity) or $|1\rangle, |3\rangle, \ldots, |n-2\rangle$ (for odd parity).

In a convenient notation we may write the mth solution of (4.1) with $E_m - \lambda_m/g = \varepsilon_{n-2}$ as

$$y^m(x) = P_m(\varepsilon_{n-2}, x) \, e^{-x^2/2}$$

and then (4.2) becomes

$$\int P_m(\varepsilon_{n-2}, x) P_{m'}(\varepsilon_{n-2}, x) \frac{e^{-x^2}}{1+gx^2} \, dx = 0 \qquad m \neq m'.$$

That is, the polynomials $P_m(\varepsilon_{n-2}, x)$ are orthogonal with respect to the weight function $e^{-x^2}/(1+gx^2)$. Unlike the standard kinds of orthogonal polynomials, however, these are not independent of the size of the space of monomials from which they are constructed. The highest power of x appearing in the $P_m(\varepsilon_{n-2}, x)$ is determined by ε_{n-2}, and there is no particular connection between polynomials corresponding to different ε_{n-2}.

To call the $P_m(\varepsilon_{n-2}, x)$ orthogonal polynomials when they are orthogonal only in the restricted sense described and when, expressed as sums of monomials, they each have, for given ε_{n-2}, the same degree of complexity may seem to be stretching a point. However, the important thing about orthogonal polynomials is not the highest power of x that a polynomial contains but the number of real zeros it possesses. In fact, ordinary orthogonal polynomials possess only real zeros. The $P_m(\varepsilon_{n-2}, x)$ on the other hand have in general both real and complex zeros. Table 1 shows the three solutions, for $g = 1$, corresponding to $\varepsilon_{n-2} = 9$. Here the $|0\rangle$, $|2\rangle$ and $|4\rangle$ oscillator states are involved. The $\lambda = 0$ solution is just the $|4\rangle$ state and it has four real zeros. The other two solutions have two real zeros and no real zeros respectively. Our polynomials thus appear to have a behaviour analogous to that of the usual orthogonal polynomials. Within a given set corresponding to a fixed ε_{n-2} the number of real roots varies in steps of two up to the maximum possible in the $\lambda = 0$ solution.

Table 1. Polynomials and λ values for the case $\varepsilon_{n-2} = 9$, $g = 1$.

λ	$P(\varepsilon_{n-2} = 9, x)$
0	$1 - 4x^2 + \frac{4}{3}x^4$
−8.88	$(1 - 0.562x^2)(1 + x^2)$
−17.12	$(1 + 3.562x^2)(1 + x^2)$

5. Summary

We have shown that polynomial solutions of the form (2.1) with $P(t)$ given by (2.13) exist for any given n. Although solutions of a generally similar character are known to exist for other non-harmonic oscillators, this is the only case we know of where general proofs can be given. We also showed that the polynomial solutions may be viewed as the Sturmians of the differential equation (4.1), and that in consequence they may be regarded as a new species of non-classical orthogonal polynomials.

Acknowledgments

We wish to thank N Rowley and M Znojil for helpful comments. RRW wishes to acknowledge the tenure of a Science Research Council Senior Fellowship during the course of this work.

References

Bessis N and Bessis G 1980 *J. Math. Phys.* **21** 2780
Biswas S N, Datta K, Saxena R P, Srivastava P K and Varma V S 1973 *J. Math. Phys.* **14** 1190
Flessas G P 1981 *Phys. Lett.* **83A** 121
Flessas G P and Das K P 1980 *Phys. Lett.* **78A** 19
Gantmacher F R 1959 *The Theory of Matrices* (New York: Chelsea)
Haken H 1970 *Laser Theory* in *Encyclopedia of Physics* vol XXV/2C (Princeton, NJ: Van Nostrand)
Magyari E 1981 *Phys. Lett.* **81A** 116
Mitra A K 1978 *J. Math. Phys.* **19** 2018
Risken H and Vollmer H D 1967 *Z. Phys.* **201** 323
Rotenberg M 1962 *Ann. Phys.* **19** 262
Singh V *et al* 1980 *Lett. Math. Phys.* **4** 131